Homework 1, due 2/1

Only your **four** best solutions will count towards your grade.

1. Consider the power series
$$f(z_1, z_2) = \sum_{k=0}^{\infty} z_1^k z_2^k.$$
   (a) Find the region in $\mathbf{C}^2$ where the power series converges.
   (b) Find an analytic continuation of f to a larger domain.

2. Let $U \subset \mathbf{C}^n$ be open, and $f : U \setminus \mathbf{C}^{n-2} \to \mathbf{C}$ be holomorphic, where $\mathbf{C}^{n-2} \subset \mathbf{C}^n$ is a coordinate subspace. Show that f extends to a holomorphic function $\tilde{f} : U \to \mathbf{C}$.

3. Let $f : \mathbf{C}^n \to \mathbf{C}$ be holomorphic, with $n > 1$. Show that $f^{-1}(0)$ cannot be contained in a bounded set of $\mathbf{C}^n$ if it is non-empty.

4. Let $f : U \to \mathbf{C}^n$, where $U \subset \mathbf{C}^m$ and $m \leq n$. Suppose that $J(f)(z_0)$ has rank m at some $z_0 \in U$. Show that there is a neighborhood V of $f(z_0)$ and a biholomorphism $h : V \to V'$ for some $V' \subset \mathbf{C}^n$ such that
$$h(f(z_1, \ldots, z_m)) = (z_1, \ldots, z_m, 0, \ldots, 0),$$
for $(z_1, \ldots, z_m) \in V$.

5. Let $p(z)$ be a polynomial with $p(0) = p'(0) = 0$. Consider the map
$$F(z_1, z_2) = (z_2, -2z_1 - p(z_2)).$$
Show that $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a biholomorphism with Jacobian
$$A = J(F)(0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$
at the origin.

6. In the notation of the previous question,
   (a) Define the set $U = \{z \in \mathbf{C}^2 \,:\, F^{-n}(z) \to 0 \text{ as } n \to \infty\}$, and show that we can define a biholomorphism $G : \mathbf{C}^2 \to U$ as follows: for z close to 0 we define $G(z)$ by the formula
$$G(z) = \lim_{n\to\infty} F^n(A^{-n}(z)),$$
   and extend G to all of $\mathbf{C}^2$ using the identity $G \circ A = F \circ G$.
   (b) Suppose that we also have $p(1) = -3, p'(1) = 0$. Show that then $(1, 1)$ is a fixed point of F and U omits a neighborhood of $(1, 1)$. So G defines a biholomorphism between $\mathbf{C}^2$ and a subset U of $\mathbf{C}^2$ omitting an open set.